

More on the metric projection onto a closed convex set in a Hilbert space

BIAGIO RICCERI

Here and in what follows, $(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space and X is a non-empty closed convex subset of H .

For each $x \in H$, we denote by $P(x)$ the metric projection of x on X , that is the unique global minimum of the restriction of the functional $y \rightarrow \|x - y\|$ to X .

There is no doubt that the map P is among the most important and studied ones within convex analysis, functional analysis and optimization theory.

For the above reason, we think that it is of interest to highlight some properties of P which do not appear in the wide literature concerning P .

We collect such properties in Theorems 1, 2 and 3 below.

First, we fix some notations.

For each $r > 0$, we put

$$B_r = \{x \in H : \|x\|^2 < r\}$$

and

$$S_r = \{x \in H : \|x\|^2 = r\} .$$

Moreover, for each $x \in X$, we set

$$J(x) = \frac{1}{2}(\|x\|^2 - \|x - P(x)\|^2 + \|P(0)\|^2) .$$

Furthermore, for each $r > 0$, we put

$$\gamma(r) = \inf_{x \in S_r} \|x - P(x)\|^2 .$$

Finally, since P is non-expansive in H , for each $\lambda \in]-1, 1[$, the map λP is a contraction and hence has a unique fixed point that we denote by \hat{y}_λ .

THEOREM 1. - *Assume that $0 \notin X$.*

Then, the following assertions hold:

- (c₁) *the function $\lambda \rightarrow g(\lambda) := J(\hat{y}_\lambda)$ is increasing in $] -1, 1[$ and its range is $] -\|P(0)\|^2, \|P(0)\|^2[$;*
- (c₂) *for each $r \in] -\|P(0)\|^2, \|P(0)\|^2[$, the point $\hat{x}_r := \hat{y}_{g^{-1}(r)}$ is the unique point of minimal norm of $J^{-1}(r)$ towards which every minimizing sequence in $J^{-1}(r)$, for the norm, converges ;*
- (c₃) *the function $r \rightarrow \hat{x}_r$ is continuous in $] -\|P(0)\|^2, \|P(0)\|^2[$;*
- (c₄) *the function $\lambda \rightarrow h(\lambda) := \|\hat{y}_{\frac{1}{\lambda}}\|^2$ is decreasing in $]1, +\infty[$ and its range is $]0, \|P(0)\|^2[$;*
- (c₅) *for each $r \in]0, \|P(0)\|^2[$, the point $\hat{v}_r := \hat{y}_{\frac{-1}{h^{-1}(r)}}$ is the unique global maximum of $J|_{S_r}$ towards which every maximizing sequence for $J|_{S_r}$ converges ;*
- (c₆) *the function $r \rightarrow \hat{v}_r$ is continuous in $]0, \|P(0)\|^2[$.*

Assuming, in addition, that X is compact, the following assertions hold:

- (c₇) *the function γ is C^1 , decreasing and strictly convex in $]0, \|P(0)\|^2[$;*
- (c₈) *one has*

$$P(\hat{v}_r) = -\gamma'(r)\hat{v}_r$$

for all $r \in]0, \|P(0)\|^2[$;

- (c₉) *one has*

$$\gamma'(r) = -h^{-1}(r)$$

for all $r \in]0, \|P(0)\|^2[$.

PROOF. Clearly, the set of all fixed points of P agrees with X . Now, fix $u \in H$ and $\lambda < 1$. We show that

$$P(u + \lambda(P(u) - u)) = P(u) . \quad (1)$$

If $u \in X$, this is clear. Thus, assume $u \notin X$ and hence $P(u) \neq u$. Let $\varphi : H \rightarrow \mathbf{R}$ be the continuous linear functional defined by

$$\varphi(x) = \langle P(u) - u, x \rangle$$

for all $x \in H$. Clearly, $\|\varphi\|_{H^*} = \|P(u) - u\|$. We have

$$\text{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u)))) = \frac{|\varphi(u + \lambda(P(u) - u)) - \varphi(P(u))|}{\|\varphi\|_{H^*}} = (1 - \lambda)\|P(u) - u\| . \quad (2)$$

Moreover, by a classical result ([6], Corollary 25.23), we have

$$\langle P(u) - u, P(u) - x \rangle \leq 0$$

for all $x \in X$, that is

$$X \subseteq \varphi^{-1}([\varphi(P(u)), +\infty[) . \quad (3)$$

Also, notice that

$$\text{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u)))) = \text{dist}(u + \lambda(P(u) - u), \varphi^{-1}([\varphi(P(u)), +\infty[)) . \quad (4)$$

Indeed, otherwise, it would exist $w \in H$, with $\varphi(w) > \varphi(P(u))$, such that

$$\|u + \lambda(P(u) - u) - w\| < \text{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u)))) .$$

Then, since $\varphi(u + \lambda(P(u) - u)) < \varphi(P(u))$ (indeed $\varphi(u + \lambda(P(u) - u)) - \varphi(P(u)) = (\lambda - 1)\|P(u) - u\|^2$), by connectedness and continuity, in the open ball centered at $u + \lambda(P(u) - u)$, of radius $\text{dist}(u + \lambda(P(u) - u), \varphi^{-1}(\varphi(P(u))))$, it would exist a point at which φ takes the value $\varphi(P(u))$, which is absurd. So, (4) holds. Now, from (2), (3), (4), it follows that

$$(1 - \lambda)\|P(u) - u\| \leq \text{dist}(u + \lambda(P(u) - u), X) \leq \|u + \lambda(P(u) - u) - P(u)\| = (1 - \lambda)\|P(u) - u\|$$

which yields (1). From (1), in particular, we infer that $P(0) = P(-P(0))$. On the other hand, if $\tilde{x} \in H$ is such that $\tilde{x} = -P(\tilde{x})$, then, applying (1) with $u = \tilde{x}$ and $\lambda = \frac{1}{2}$, we get $P(0) = P(\tilde{x})$ and so $\tilde{x} = -P(0)$. Therefore, $-P(0)$ is the unique fixed point of $-P$. Now, let us recall that J is a Fréchet differentiable convex functional whose derivative is equal to P ([1], Proposition 2.2). This allows us to use the results of [3]. Therefore, (c_1) , (c_2) , (c_3) follow respectively from (a_1) , (a_2) , (a_3) of Theorem 3.2 of [3], since (with the notation of that result) we have $\eta_1 = J(-P(0)) = -\|P(0)\|^2$ and $\theta_1 = \inf_X J = \|P(0)\|^2$, while (c_4) , (c_5) , (c_6) follow respectively from (b_1) , (b_2) , (b_3) of Theorem 3.3 of [3], since $\theta_2 = \|P(0)\|^2$. Now, assume that X is also compact. Then, J turns out to be sequentially weakly continuous ([5], Corollary 41.9). Moreover, J has no local maxima since P has no zeros. At this point, (c_7) , (c_8) , (c_9) follow respectively from (b_4) , (b_5) , (b_6) of Theorem 3.3 of [3], since, for a constant k_0 , we have

$$\sup_{S_r} J = -\frac{1}{2}\gamma(r) + k_0$$

for all $r > 0$. The proof is complete. \triangle

THEOREM 2. - Let $Q : H \rightarrow H$ be a continuous and monotone potential operator such that

$$\lim_{\|x\| \rightarrow +\infty} I(x) := \int_0^1 \langle Q(sx), x \rangle ds = +\infty .$$

Set

$$\lambda^* = \inf_{r > \inf_H I} \inf_{x \in I^{-1}([-\infty, r])} \frac{J(x) - \inf_{y \in I^{-1}([-\infty, r])} J(y)}{r - I(x)}.$$

Then, the equation

$$P(x) + \lambda Q(x) = 0$$

has a solution in H for every $\lambda > \lambda^*$. Moreover, when $\lambda^* > 0$, the same equation has no solution in H for every $\lambda < \lambda^*$.

PROOF. Since Q is a monotone potential operator, the functional I turns out to be convex, of class C^1 and its derivative agrees with Q . Now, the conclusion follows from Theorem 2.4 of [2], since, by convexity, the solutions of the equation $P(x) + \lambda Q(x)$ are exactly the global minima in H of the functional $J + \lambda I$. \triangle

THEOREM 3. - Let (T, \mathcal{F}, μ) be a measure space, with $0 < \mu(T) < +\infty$ and assume that $0 \notin X$. Then, for every $\eta \in L^\infty(T)$, with $\eta \geq 0$, for every $r \in]0, \|P(0)\|^2[$ and for every $p \geq 2$, if we put

$$U_{\eta, r} = \left\{ u \in L^p(T, H) : \int_T \eta(t) \|u(t)\|^2 d\mu = r \int_T \eta(t) d\mu \right\},$$

we have

$$\inf_{u \in U_{\eta, r}} \int_T \eta(t) \|u(t) - P(u(t))\|^2 d\mu = \inf_{x \in S_r} \|x - P(x)\|^2 \int_T \eta(t) d\mu \quad (5)$$

and

$$\sup_{u \in U_{\eta, r}} \int_T \eta(t) \|u(t) - P(u(t))\|^2 d\mu = \sup_{x \in S_r} \|x - P(x)\|^2 \int_T \eta(t) d\mu. \quad (6)$$

PROOF. Applying Theorem 5 of [4] to J and $-J$, respectively, we obtain

$$\inf_{u \in V_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu = \inf_{S_r} J \int_T \eta(t) d\mu \quad (7)$$

and

$$\sup_{u \in V_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu = \sup_{S_r} J \int_T \eta(t) d\mu, \quad (8)$$

where

$$V_{\eta, r} = \left\{ u \in L^p(T, H) : \int_T \eta(t) \|u(t)\|^2 d\mu \leq r \int_T \eta(t) d\mu \right\}.$$

Now, observe that $J|_{S_r}$ has a global minimum. Indeed, since J is weakly lower semicontinuous and $\overline{B_r}$ is weakly compact, $J|_{\overline{B_r}}$ has a global minimum, say \hat{w}_r . Notice that $\hat{w}_r \in S_r$, since, otherwise, $P(\hat{w}_r) = 0$ which is impossible since $0 \notin X$. So, \hat{w}_r is a global minimum of $J|_{S_r}$. Furthermore, from Theorem 1, we know that $J|_{S_r}$ has a global maximum, say \hat{v}_r . Denote by the same symbols the constant functions (from T into Y) taking, respectively, the values \hat{w}_r and \hat{v}_r . Since $\mu(T) < +\infty$, we have $\hat{w}_r, \hat{v}_r \in U_{\eta, r}$. So, from (7) and (8), it follows respectively

$$\inf_{u \in V_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu = \int_T J(\hat{w}_r) \eta(t) d\mu \geq \inf_{u \in U_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu$$

and

$$\sup_{u \in V_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu = \int_T J(\hat{v}_r) \eta(t) d\mu \leq \sup_{u \in U_{\eta, r}} \int_T \eta(t) J(u(t)) d\mu.$$

Therefore

$$\inf_{S_r} J \int_T \eta(t) d\mu = (r + \|P(0)\|^2 - \sup_{x \in S_r} \|x - P(x)\|^2) \int_T \eta(t) d\mu = \inf_{u \in U_{\eta, r}} \int_T \eta(t) (\|u(t)\|^2 - \|u(t) - P(u(t))\|^2 + \|P(0)\|^2) d\mu$$

$$= (r + \|P(0)\|^2) \int_T \eta(t) d\mu - \sup_{u \in U_{\eta,r}} \int_T \eta(t) \|u(t) - P(u(t))\|^2 d\mu$$

which yields (6). Likewise

$$\begin{aligned} \sup_{S_r} J \int_T \eta(t) d\mu &= (r + \|P(0)\|^2 - \inf_{x \in S_r} \|x - P(x)\|^2) \int_T \eta(t) d\mu = \sup_{u \in U_{\eta,r}} \int_T \eta(t) (\|u(t)\|^2 - \|u(t) - P(u(t))\|^2 + \|P(0)\|^2) d\mu \\ &= (r + \|P(0)\|^2) \int_T \eta(t) d\mu - \inf_{u \in U_{\eta,r}} \int_T \eta(t) \|u(t) - P(u(t))\|^2 d\mu \end{aligned}$$

which yields (5)

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References

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Department of Mathematics
University of Catania
Viale A. Doria 6
95125 Catania, Italy
e-mail address: ricceri@dmf.unict.it